Name: _____ Class / Date: _____

ANSWER VERSION. Simulated values vary from run to run — with 1,000 samples accept means within about ±0.01 (proportions) and SDs within about ±10%.

1 Simulator 1 — Sample proportion $\hat{p}$

A. Center

SIMULATE — Run each row and record the mean of the 1,000 values of $\hat{p}$.

Run	p	n	Simulated mean of $\hat{p}$	Run	p	n	Simulated mean of $\hat{p}$
1	0.30	20	≈ 0.30	3	0.70	50	≈ 0.70
2	0.30	100	≈ 0.30	4	0.10	200	≈ 0.10

a Compare Runs 1 and 2. Did increasing *n* move the center? What does decide where the center is?

No — both centered at ≈ 0.30; larger *n* only made it narrower. The population proportion *p* decides the center.

RULE (center): $\mu_{\hat{p}}$ = *p* So $\hat{p}$ is *unbiased*, which means: on average over many samples, $\hat{p}$ equals *p*

B. Spread

SIMULATE — Keep $p = 0.50$. Record the SD of $\hat{p}$ for each *n*.

n	25	100	400	When *n* is multiplied by 4, the SD is …
Simulated SD of $\hat{p}$	≈ 0.100	≈ 0.050	≈ 0.025	multiplied by ½ (halved)

SIMULATE — Now keep $n = 100$ and change *p*. Fill in the last two rows by calculator.

p	0.10	0.30	0.50	0.70	0.90
Simulated SD of $\hat{p}$	≈ 0.030	≈ 0.046	≈ 0.050	≈ 0.046	≈ 0.030
p(1 − p)	0.09	0.21	0.25	0.21	0.09
$\sqrt{\frac{p(1-p)}{n}}$	0.030	0.046	0.050	0.046	0.030

b Where is the SD largest? Where is it smallest? Which two values of *p* gave the same SD, and why?

Largest at $p = 0.50$, smallest at $p = 0.10$ and 0.90. Pairs *p* and $1 - p$ (0.30 & 0.70, 0.10 & 0.90) give the same SD because p(1 − p) is the same product.

RULE (spread), written as a fraction: $\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$ To cut the SD in half, multiply *n* by 4 .

C. The 10% Condition

Real samples are drawn **without replacement**. The SD rule above is only accurate when the sample is a small part of the population: **n ≤ 0.10 N** (N = population size). Always compare *n* with **0.10 N**, not with N.

Situation	n	N	0.10 N	Pass?
Poll of a school	60	500	50	No (60 > 50)
Survey of a city	400	80,000	8,000	Yes

c If the 10% Condition fails, can you still calculate $\hat{p}$? Which number can you no longer trust?

Yes, $\hat{p}$ = X/n can still be calculated; the SD formula can no longer be trusted.

D. Shape — how the population (p) shapes the histogram

SIMULATE — For each cell write the shape: **S** = symmetric, **R** = right-skewed, **L** = left-skewed; add **N** if it looks Normal (bell-shaped). Also fill in the two counts.

	n = 10	n = 40	n = 160
p = 0.50 (balanced)	shape: **S** np = **5** n(1−p) = **5**	shape: **S, N** np = **20** n(1−p) = **20**	shape: **S, N** np = **80** n(1−p) = **80**
p = 0.10 (lopsided)	shape: **R** np = **1** n(1−p) = **9**	shape: **R** np = **4** n(1−p) = **36**	shape: **S, N** np = **16** n(1−p) = **144**
p = 0.90 (lopsided)	shape: **L** np = **9** n(1−p) = **1**	shape: **L** np = **36** n(1−p) = **4**	shape: **S, N** np = **144** n(1−p) = **16**

d When $p = 0.10$ and n is small, which way is the histogram skewed? Explain using the fact that $\hat{p}$ can never be below 0.

Right-skewed. Most values of $\hat{p}$ pile up near 0.10, close to the "wall" at 0 — $\hat{p}$ cannot go below 0, so the only room for a long tail is to the right.

e Look at all the cells that looked Normal. What do their two counts have in common? Complete the rule.

In every Normal-looking cell, both np and n(1 − p) are at least 10.

RULE (Large Counts): $\hat{p}$ is approximately Normal when **np** ≥ **10** and **n(1−p)** ≥ **10**. Shape depends on ***p*** and ***n*** together.

2 Simulator 2 — Difference of proportions $\hat{p}_1 - \hat{p}_2$

A. Center

SIMULATE — Use $n_1 = n_2 = 50$. Record the mean of the 1,000 differences.

p_1	p_2	$p_1 - p_2$	Simulated mean of $\hat{p}_1 - \hat{p}_2$
0.60	0.35	**0.25**	**≈ 0.25**
0.30	0.55	**−0.25**	**≈ −0.25**

RULE (center): $\mu_{\hat{p}_1 - \hat{p}_2}$ = **$p_1 - p_2$** A negative center means: **population 2 has the larger proportion ($p_2 > p_1$)**

B. Spread — the surprise

SIMULATE — Set $p_1 = p_2 = 0.50$ and $n_1 = n_2 = 100$.

① SD of each $\hat{p}$ alone (from Simulator 1): $\sigma_{\hat{p}_1} = \sigma_{\hat{p}_2}$ = **0.050**

② Predict the SD of the difference — circle: 0 / 0.050 / **(0.071)** / 0.100 ③ Simulated SD of $\hat{p}_1 - \hat{p}_2$: **≈ 0.071**

④ Test three guesses: $\sigma_1 - \sigma_2$ = **0** $\sigma_1 + \sigma_2$ = **0.100** $\sqrt{\sigma_1^2 + \sigma_2^2}$ = **0.0707 ✓** ← which one matches ③?

f Why is the difference **more** variable than either sample on its own, even though we subtract? (Think: how many random things are moving?)

Two random things are moving: $\hat{p}_1$ and $\hat{p}_2$ both vary. Their chance errors can push the difference in the same direction, so the variability adds up — it never cancels.

RULE (spread): variances **add**. $\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$

g Check your rule: $p_1 = 0.45$, $n_1 = 80$, $p_2 = 0.30$, $n_2 = 120$. Calculate the mean and SD of $\hat{p}_1 - \hat{p}_2$ (both fractions under one square root), then compare with the simulator.

Mean = 0.45 − 0.30 = 0.15. SD = $\sqrt{\frac{0.45 \cdot 0.55}{80} + \frac{0.30 \cdot 0.70}{120}} = \sqrt{0.003094 + 0.00175} = \sqrt{0.004844} \approx 0.070.$

Simulator ≈ 0.15 and ≈ 0.070 ✓

C. Conditions and shape for a difference

SIMULATE — (i) $p_1 = 0.05$, $n_1 = 20$ with $p_2 = 0.50$, $n_2 = 200$. (ii) Same *p*'s, but $n_1 = 200$. Sketch both histograms of $\hat{p}_1 - \hat{p}_2$.

(i) $n_1 = 20$, $n_2 = 200$

Lumpy and right-skewed: $\hat{p}_1$ can only be 0, 0.05, 0.10, … so the difference comes in separate clumps with a long right tail.

(ii) $n_1 = n_2 = 200$

Smooth, symmetric, bell-shaped — approximately Normal, centered at −0.45.

Four counts	n_1p_1	$n_1(1-p_1)$	n_2p_2	$n_2(1-p_2)$	All ≥ 10?
(i)	1	19	100	100	No
(ii)	10	190	100	100	Yes

h In (i), which sample spoiled the Normal shape? How many counts must you check for a difference, and what does one failing count do?

Sample 1 ($n_1p_1 = 1 < 10$). Check all four counts — one failing count is enough to make the difference not Normal.

CONDITIONS for $\hat{p}_1 - \hat{p}_2$: 10% Condition for **each (both)** sample(s); the two samples must be **independent** ; Large Counts: all **four** counts ≥ 10.

3 Simulator 3 — Sample mean $\bar{x}$

Now each value is a number (height, score…). The population has mean μ, SD σ and a **shape** you choose in the simulator.

A. Center and spread

SIMULATE — Choose the **right-skewed** population. Copy the population values: μ = **30** σ = **30** . Then run each *n*.

n	Simulated mean of $\bar{x}$	Simulated SD of $\bar{x}$	$\frac{\sigma}{\sqrt{n}}$ (calculator)	Close to the simulated SD?
5	≈ 30	≈ 13.4	13.42	Yes
20	≈ 30	≈ 6.7	6.71	Yes
50	≈ 30	≈ 4.2	4.24	Yes

a Did *n* change the center? When *n* went from 5 to 20 (× 4), what happened to the SD? Is that the same pattern as for $\hat{p}$?

No — the center stayed at μ = 30. From *n* = 5 to 20 the SD was cut in half (13.4 → 6.7). Yes, the same $\frac{1}{\sqrt{n}}$ pattern as $\hat{p}$.

RULES: $\mu_{\bar{x}}$ = **μ** $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ (use only if the **10%** Condition holds)

b Check: μ = 72, σ = 12, n = 36 from a population of N = 300. Find $\mu_{\bar{x}}$ and $\sigma_{\bar{x}}$. Is the SD formula allowed here? Show 0.10 N.

$\mu_{\bar{x}} = 72$, $\sigma_{\bar{x}} = \frac{12}{\sqrt{36}} = 2$. But 0.10 N = 0.10 × 300 = 30 and 36 > 30, so the 10% Condition fails — the SD formula is not allowed .

B. Shape — how the population shape affects the histogram of $\bar{x}$

SIMULATE — For each population shape run $n = 2, 10, 30$. Write **P** = still looks like the population, **B** = in between, **N** = Normal-looking.

Population shape	n = 2	n = 10	n = 30
Normal	N	N	N
Uniform	B (triangle)	N	N
Right-skewed	P	B (still skewed)	N (roughly)
Bimodal	P / B (bumpy)	B	N

c Which population gave a Normal-looking $\bar{x}$ even at $n = 2$? Which needed the largest n? Did the population shape change the *center* or the *SD* of $\bar{x}$, or only its *shape*?

The Normal population — $\bar{x}$ was Normal at every n. The right-skewed population needed the largest n. Population shape changed only the **shape** of $\bar{x}$: the center was always μ and the SD was always $\frac{\sigma}{\sqrt{n}}$.

RULE (shape): Normal population → $\bar{x}$ is Normal for any (every) n. Other shapes → approx. Normal when $n \geq$ 30 (the Central Limit Theorem). For $\hat{p}$, "population shape" means the value of p.

4 Simulator 4 — Difference of means $\bar{x}_1 - \bar{x}_2$

SIMULATE — Population 1: Normal, $\mu_1 = 50$, $\sigma_1 = 10$, $n_1 = 40$. Population 2: Normal, $\mu_2 = 42$, $\sigma_2 = 8$, $n_2 = 40$. Predict first, then simulate.

	My prediction (show the calculation)	Simulator
Mean of $\bar{x}_1 - \bar{x}_2$	$50 - 42 = 8$	≈ 8
SD of $\bar{x}_1 - \bar{x}_2$	$\sqrt{\frac{100}{40} + \frac{64}{40}} = \sqrt{2.5 + 1.6} = \sqrt{4.1} \approx 2.02$	≈ 2.0

RULES: $\mu_{\bar{x}_1 - \bar{x}_2} =$ $\mu_1 - \mu_2$ $\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$ means subtract, variances add

SIMULATE — Make both populations **right-skewed**. Run (i) $n_1 = 5$, $n_2 = 50$, then (ii) $n_1 = n_2 = 50$.

d Describe the shape of $\bar{x}_1 - \bar{x}_2$ in (i) and (ii). Which sample size controlled the shape in (i)? Which condition explains the change?

(i) Still right-skewed — the small sample $n_1 = 5$ controls the shape. (ii) Approximately Normal, because both $n \geq 30$ (Central Limit Theorem).

Summary — fill in from your own results

Statistic	Mean	SD (if 10% Condition holds)	Approximately Normal when …
$\hat{p}$	p	$\sqrt{\frac{p(1-p)}{n}}$	$np \geq 10$ and $n(1-p) \geq 10$
$\hat{p}_1 - \hat{p}_2$	$p_1 - p_2$	$\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$	all four counts ≥ 10
$\bar{x}$	μ	$\frac{\sigma}{\sqrt{n}}$	population Normal, or $n \geq 30$ (CLT)
$\bar{x}_1 - \bar{x}_2$	$\mu_1 - \mu_2$	$\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$	both populations Normal, or both $n \geq 30$